Math 213 • Final exam • May 13, 2014

1. **(10 pts)** Consider an object moving with acceleration $a(t)=\left\langle 3\cos^2 t\,\sin t,\ \dfrac{t^3}{t^2+1}\right\rangle$. Find its velocity if the initial velocity is $\mathbf{v}(0)=\langle 0,\ 2\rangle$.

$$a(t)=\left\langle 3\cos^2 t\,\sin t,\ \frac{t^3}{\underbrace{t^2+1}_{u}}\right\rangle$$

$$\Rightarrow\ v(t)=\left\langle 3\int\cos^2 t\,\underbrace{\sin t\,dt}_{-d\cos t},\ \ \frac{1}{2}\int\frac{(u-1)\,du}{u}\right\rangle=\left\langle -\cos^3 t,\ \frac{1}{2}\left[u-\ln u\right]_{u=t^2+1}\right\rangle+\mathbf{C}$$

$$=\left\langle -\cos^3 t,\ \frac{t^2+1-\ln\left(t^2+1\right)}{2}\right\rangle+\mathbf{C}$$

Check initial condition:

$$v(0)=\left\langle -1,\ \frac{1}{2}\right\rangle+\mathbf{C}=\langle 0,\ 2\rangle\quad\Rightarrow\ \mathbf{C}=\langle 0,\ 2\rangle-\left\langle -1,\ \frac{1}{2}\right\rangle=\left\langle 1,\ \frac{3}{2}\right\rangle$$

$$\Rightarrow\ v(t)=\left\langle 1-\cos^3 t,\ \frac{3}{2}+\frac{t^2+1-\ln\left(t^2+1\right)}{2}\right\rangle=\boxed{\left\langle 1-\cos^3 t,\ 2+\frac{t^2}{2}-\frac{1}{2}\ln\left(t^2+1\right)\right\rangle}$$

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2. **(10 pts)** Find the limit, or show that it does not exist: **(a)** $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2 y}{x+y}$ **(b)** $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2-y^2}{\sin(x+y)}$

$\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2 y}{x+y}$ – does not exist, since there is a path leading to zero where the expression is infinite: $x=-y$

$$\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{\sin(x+y)}=\lim_{(x,y)\to(0,0)}\frac{(x-y)(x+y)}{\sin(x+y)}=\underbrace{\lim_{(x,y)\to(0,0)}(x-y)}_{0}\ \underbrace{\lim_{(x,y)\to(0,0)}\frac{(x+y)}{\sin(x+y)}}_{\lim\limits_{z\to 0}\frac{z}{\sin z}=1}=\boxed{0}$$

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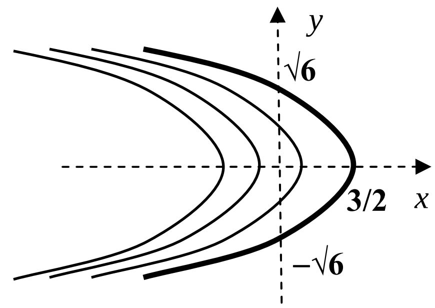
3. **(14 pts)** Consider the scalar field $f(x,y)=\sqrt{6-4x-y^2}$. Find the domain and the range of this field and sketch the domain along with any two of its level curves. Then, use the linear approximation to estimate $f(1.04,\ 0.98)$. Finally, find the rate of change of this function at point (1,1) in the direction of vector $\mathbf{u}=2\mathbf{i}+\mathbf{j}$

Domain: $6-4x-y^2\ge 0 \ \Rightarrow\ x\le\dfrac{6-y^2}{4}\ \Rightarrow\ \boxed{x\le\dfrac{3}{2}-\dfrac{y^2}{4}}$

Range: $[0;\ +\infty)$

Level Curves: $\quad f=0:\ \ x=\dfrac{3}{2}-\dfrac{y^2}{4}\ \ \left(y=\pm\sqrt{6-4x}\right)$

$f=1:\ \ x=\dfrac{5}{4}-\dfrac{y^2}{4}\ \ \left(y=\pm\sqrt{5-4x}\right)$



Linear approximation: choose point (1, 1), which is close to (1.04, 0.98)

$$\left.\begin{aligned} &f(1,1)=1\\ &f_x=\frac{-4}{2\sqrt{6-4x-y^2}}=-\frac{2}{\sqrt{6-4x-y^2}}\ \Rightarrow\ f_x(1,1)=-2\\ &f_x=\frac{-2y}{2\sqrt{6-4x-y^2}}=-\frac{y}{\sqrt{6-4x-y^2}}\ \Rightarrow\ f_x(1,1)=-1 \end{aligned}\right\}\ \Rightarrow\ \boxed{f(x,y)\approx 1-2(x-1)-(y-1)}$$

$$\Rightarrow\ f(1.04,0.98)\approx 1-2(1.04-1)-(0.98-1)=1-0.08+0.02=\boxed{0..94}$$

Rate of change of this function at point (1,1) in the direction of vector $\mathbf{u}=2\mathbf{i}+\mathbf{j}$:

Use the value of the gradient that we just calculated above:

$$D_u f(1,1)=\nabla f(1,1)\cdot\hat{\mathbf{u}}=\langle -2,-1\rangle\cdot\frac{\langle 2,\ 1\rangle}{\sqrt{2^2+1^2}}=-\frac{5}{\sqrt{5}}=\boxed{-\sqrt{5}}$$

4. **(12 pts)** Find local maxima, minima and saddle points of function $f = e^x\left(x^2 - y^2\right)$

First, let's find the critical points $(\nabla f = 0)$:

$$\left.\begin{array}{l} f_x = e^x\left(x^2-y^2\right) + 2\,x\,e^x = e^x\left(2x + x^2 - y^2\right) = 0 \quad \Rightarrow \quad x(2+x) = 0 \\ f_y = -2\,y\,e^x = 0 \;\Rightarrow\; y = 0 \end{array}\right\} \Rightarrow \; C.P.: \boxed{(0,0) \text{ and } (-2,0)}$$

Now, let's find the second order derivatives:

$$\begin{cases} f_{xx} = e^x\left(2x + x^2 - y^2\right) + e^x(2+2x) = e^x\left(2 + 4x + x^2 - y^2\right) \\ f_{yy} = -2\,e^x \\ f_{xy} = -2\,y\,e^x \end{cases}$$

$$(0,0): \begin{cases} f_{xx} = 2 \\ f_{yy} = -2 \\ f_{xy} = 0 \end{cases} \Rightarrow \; H = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0 \;\Rightarrow\; (0,0) \text{ is a saddle point}$$

$$(-2,0): \begin{cases} f_{xx} = e^{-2}(2 - 8 + 4) = -2e^{-2} < 0 \\ f_{yy} = -2e^{-2} < 0 \\ f_{xy} = 0 \end{cases} \Rightarrow \; H = \begin{vmatrix} -2e^{-2} & 0 \\ 0 & -2e^{-2} \end{vmatrix} = 4e^{-4} > 0 \;\Rightarrow\; (-2,0) \text{ is a local max}$$

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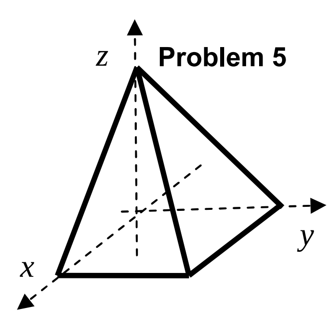
5. **(14 pts)** Find the mass of the solid of density δ=*y* enclosed in the first octant by the planes $x + z = 1$ and $y + z = 1$, as shown in the figure. Which orders of integration *cannot* be used without breaking apart the integral into several pieces?



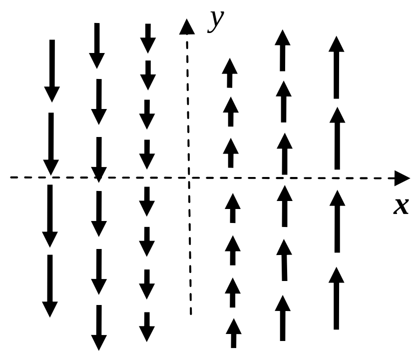
$$M = \iiint \delta(\mathbf{r})\, dV = \int_0^1 \int_0^{1-z} \int_0^{1-z} y\, dy\, dx\, dz = \frac{1}{2}\int_0^1 \int_0^{1-z} (1-z)^2\, dx\, dz$$

$$= \frac{1}{2}\int_0^1 (1-z)^3\, dz = -\frac{1}{8}(1-z)^4\Big|_0^1 = \boxed{\frac{1}{8}}$$

The following orders are *not* convenient: $dz\,dx\,dy$, $dz\,dy\,dx$

6. **(14 pts)** Sketch the vector field $\vec{\mathbf{F}}(x,y,z)=\langle 0,\ x\rangle$, and verify the Green's Theorem in circulation-curl form for this vector field and the region between the curves $x+2y=0$ and $x=3-y^2$, by calculating both integrals that appear in this theorem. Make sure to sketch the region of integration

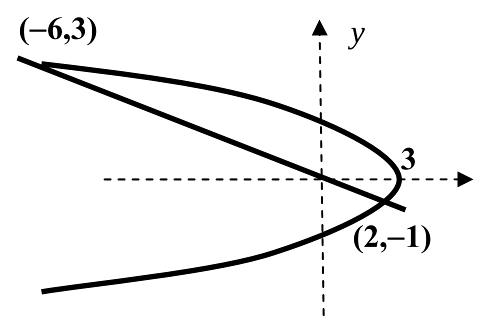
Sketch of the vector field:



First, find the intersections of the two curves:

$$x=3-y^2=-2y \;\Rightarrow\; y^2-2y-3=0 \;\Rightarrow\; (y-3)(y+1)=0 \;\Rightarrow\; (2,\,-1) \text{ and } (-6,3)$$

$$\oint_{\partial R}(M\,dx+N\,dy)=\iint_R \underbrace{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}_{=1}dx\,dy$$



Let's calculate the integral of circulation density first:

$$\iint_R dx\,dy=\int_{-1}^{3}\int_{-2y}^{3-y^2}dx\,dy=\int_{-1}^{3}\left(3-y^2+2y\right)dy=\left[3y-\frac{y^3}{3}+y^2\right]_{-1}^{3}$$

$$=(9-9+9)-\left(-3+\frac{1}{3}+1\right)=9+2-\frac{1}{3}=\boxed{\frac{32}{3}}$$

Now, the total circulation of **F** along the closed curve:

$$\oint_{\partial R}(M\,dx+N\,dy)=\oint_{\partial R}x\,dy=\int_{\substack{\text{BOTTOM:}\\ x=-2y}}xdy+\int_{\substack{\text{TOP:}\\ x=3-y^2}}xdy=-2\int_{3}^{-1}y\,dy+\int_{-1}^{3}\left(3-y^2\right)dy$$

$$=\left[-y^2\right]_3^{-1}+\left[3y-\frac{y^3}{3}\right]_{-1}^{3}=(9-1)+\left(9-9+3-\frac{1}{3}\right)=8+3-\frac{1}{3}=\boxed{\frac{32}{3}}\quad \text{Verified!}$$

7. **(10 pts)** Find the total area of the conical surface $z^2 = x^2 + y^2$ enclosed between the planes $z = 0$ and $z = 4$, using surface parametrization in polar variables *r* and θ.

$$\mathbf{r} = \langle x, y, z \rangle = \langle r\cos\theta,\ r\sin\theta,\ r \rangle, \quad 0 \le \theta \le 2\pi, \quad 0 \le r \le 2$$

$$\left.\begin{aligned} \mathbf{r}_r &= \langle \cos\theta, \quad \sin\theta, \quad 1 \rangle \\ \mathbf{r}_\theta &= \langle -r\sin\theta,\ r\cos\theta,\ 0 \rangle \end{aligned}\right\} \ (\mathbf{r}_r \times \mathbf{r}_\theta) = \langle -r\cos\theta, -r\sin\theta, \mathrm{r} \rangle$$

$$d\sigma = |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}\ dr\, d\theta = r\sqrt{2}\ dr\, d\theta$$

$$A = \iint d\sigma = \sqrt{2}\int_0^{2\pi}\int_0^4 r\, dr\, d\theta = \sqrt{2}\, 2\pi \left[\frac{r^2}{2}\right]_0^4 = \boxed{16\sqrt{2}\,\pi}$$

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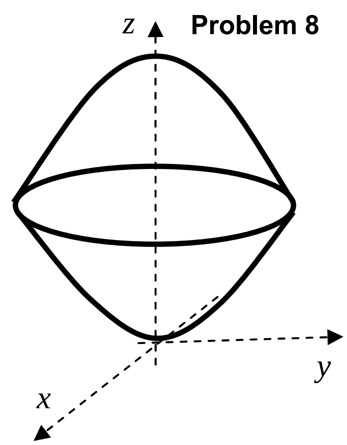
8. **(16 pts)** Verify the Divergence Theorem for the vector field $\vec{\mathbf{F}} = \langle 0, 0, z \rangle$ and the region enclosed between the surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$, by calculating the flux of $\vec{\mathbf{F}}$ across the boundary of this region, in the outward direction, and comparing the result with the volume integral of the divergence of $\vec{\mathbf{F}}$. Make sure to sketch the region of integration.

Problem 8



Divergence Theorem: $\oiint_{\partial V} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V \nabla \cdot \mathbf{F}\ dV$

First, let's find the volume integral of divergence (flux density):

$$\iiint_V \underbrace{\nabla \cdot \mathbf{F}}_{=1}\ dV = \iiint_V dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} dz\ r\, dr\, d\theta = 2\pi\int_0^1 [z]_{r^2}^{2-r^2}\ r\, dr = 2\pi\int_0^1 (2 - 2r^2)\, r\, dr$$

$$= 4\pi\int_0^1 (r - r^3)\, dr = 4\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = 4\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \boxed{\pi}$$

Compare with the flux of this field across this surface, which consists of two pieces:

$$\iint_{\text{TOP}} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \left|\begin{array}{c} \mathbf{n}\,d\sigma = \pm\dfrac{\nabla g}{\nabla g\cdot\mathbf{k}}\,dx\,dy = \dfrac{\langle 2x,\,2y,\,1\rangle}{1}\,dx\,dy \\ \vec{\mathbf{F}}\cdot\mathbf{n}\,d\sigma = z\,dx\,dy = \left(2-r^2\right)r\,dr\,d\theta \end{array}\right| = \int_0^{2\pi}\int_0^1 \left(2r-r^3\right)dr\,d\theta = 2\pi\left[r^2-\frac{r^4}{4}\right]_0^1 = \boxed{\frac{3\pi}{2}}$$

$$\iint_{\text{BOTTOM}} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \left|\begin{array}{c} \mathbf{n}\,d\sigma = \pm\dfrac{\nabla g}{\nabla g\cdot\mathbf{k}}\,dx\,dy = \dfrac{\langle 2x,\,2y,\,-1\rangle}{1}\,dx\,dy \\ \vec{\mathbf{F}}\cdot\mathbf{n}\,d\sigma = -z\,dx\,dy = -r^2\,r\,dr\,d\theta = -r^3\,dr\,d\theta \end{array}\right| = -\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = -2\pi\left[\frac{r^4}{4}\right]_0^1 = \boxed{-\frac{\pi}{2}}$$

$$\oiint_{\partial V} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_{\text{TOP}} \mathbf{F}\cdot\mathbf{n}\,d\sigma + \iint_{\text{BOTTOM}} \mathbf{F}\cdot\mathbf{n}\,dx\,dy = \frac{3\pi}{2}-\frac{\pi}{2} = \boxed{\pi} \quad \text{Verified!}$$

Extra credit (6 pts) Calculate the following derivatives, where $\vec{\mathbf{r}} = \langle x, y, z\rangle$ denotes the position vector, and $\rho = |\vec{\mathbf{r}}| = \sqrt{x^2+y^2+z^2}$ denotes its length (distance from the origin).

a) $\vec{\nabla}(\ln\rho)$ **Since $\ln\rho$ is a scalar field, this derivative is a gradient of $\ln\rho = \frac{1}{2}\ln\left(x^2+y^2+z^2\right)$:**

$$\vec{\nabla}(\ln\rho) = \left\langle \frac{\partial\ln\rho}{\partial x}, \frac{\partial\ln\rho}{\partial y}, \frac{\partial\ln\rho}{\partial z}\right\rangle = \left\langle \frac{1}{2}\frac{\partial\ln\left(x^2+y^2+z^2\right)}{\partial x}, \frac{1}{2}\frac{\partial\ln\left(x^2+y^2+z^2\right)}{\partial y}, \frac{1}{2}\frac{\partial\ln\left(x^2+y^2+z^2\right)}{\partial z}\right\rangle$$

$$= \left\langle \frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right\rangle = \frac{\langle x,y,z\rangle}{x^2+y^2+z^2} = \boxed{\frac{\vec{\mathbf{r}}}{\rho^2}}$$

b) $\vec{\nabla}\cdot\left(\dfrac{\vec{\mathbf{r}}}{\rho^2}\right)$ **Since $\dfrac{\vec{\mathbf{r}}}{\rho^2}$ is a vector field, this derivative is a divergence of** $\dfrac{\vec{\mathbf{r}}}{\rho^2} = \dfrac{\langle x,y,z\rangle}{x^2+y^2+z^2}$

$$\vec{\nabla}\cdot\left(\frac{\vec{\mathbf{r}}}{\rho^2}\right) = \frac{\partial}{\partial x}\frac{x}{x^2+y^2+z^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2+z^2} + \frac{\partial}{\partial z}\frac{y}{x^2+y^2+z^2}$$

$$= \frac{x^2+y^2+z^2-2x^2}{\left(x^2+y^2+z^2\right)^2} + \frac{x^2+y^2+z^2-2y^2}{\left(x^2+y^2+z^2\right)^2} + \frac{x^2+y^2+z^2-2z^2}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{3x^2+3z^2+3z^2-2\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^2} = \frac{x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2} = \frac{1}{x^2+y^2+z^2} = \boxed{\frac{1}{\rho^2}}$$